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International Journal of Heat and Mass Transfer 51 (2008) 3309–3324

International Journal of HEAT and MASS TRANSFER

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Analysis of the 1-D heat conduction problem for a single fin with temperature dependent heat transfer coefficient: Part I – Extended inverse and direct solutions

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Received 13 September 2007; received in revised form 22 October 2007
Available online 11 February 2008

Abstract

A closed-form inverse solution of the 1-D heat conduction problem for a single fin or spine of constant cross section with an insulated tip is generalized to account for the effect of the tip heat loss. The heat transfer coefficient (HTC) is assumed to exhibit the power-law type dependence on the local excess temperature with arbitrary value of the exponent n in the range of $-0.5 \leqslant n \leqslant 5$. The form of the obtained inverse solution is the same as the one for a fin with an insulated tip. However, in addition to the dimensionless fin tip temperature T_e and n, the fin parameter N also depends on the complex parameter $\omega^2 Bi$. Using the inversion of this solution and a linearization procedure, the recurrent direct solution with a high convergence rate is derived. Based on the latter, the explicit direct closed-form solution for the accurate determination of the temperature distribution along a fin height at the given values of N, n, and $\omega^2 Bi$ is obtained. This allows one to determine the base thermal conductance G of the straight plate fin (SPF) and cylindrical pin fin (CPF). The relations between the fin parameters are systematized and collected in two tables for the SPF and CPF. They permit one to determine the arbitrary dimensionless geometrical or thermal fin parameter at given value of any other of its parameters and prescribed or calculated values of the main fin parameter(s) N or (and) G.

Keywords: Single fin; Temperature dependent HTC; Fin tip heat loss; 1-D heat conduction problem; Inverse and direct solutions; Relations between fin parameters

1. Introduction

This paper develops an analytical approach proposed in [1,2] to solve the 1-D nonlinear heat conduction problem for a single straight fin of constant cross section governed by the power-law type dependence of the local heat transfer coefficient on the temperature difference between the fin surface and environment. Such problems are often considered in thermal design of fins and finned surfaces with a non-uniform heat transfer coefficient (see, for example, review [3] and books [4,5]).

It is well known that in the most practical applications the local heat transfer coefficient h along the fin height is not uniform. It depends on the local coordinate or the temperature difference between the fin surface and environment. The latter dependence is usually expressed as a power function of ϑ, $h = a\vartheta^n$ with constant a and n. The 1-D steady-state problem of heat conduction, governed by the power-law type temperature dependent local heat transfer coefficient for a single straight fin of uniform cross section is reduced to the solution of ordinary nonlinear second-order differential equation with two boundary conditions. The analytical and numerical solutions of such problems for the definite values of n made their appearance in the middle 1960s– early 1970s (see, for example, papers [6–9]). Ünal [10–12] showed that the explicit analytical closed-form solutions can be obtained only for several definite values of $n = -4$, -1, and for $n = 0$. Latter corresponds to the uniform heat transfer coefficient over the

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doi:10.1016/j.ijheatmasstransfer.2007.11.027

Nomenclature

a, a_e given constants in the heat transfer equation for the lateral surfaces and tip surface of a fin ($\mathrm{W\,m^{-2}\,K^{-(n+1)}}$)
a_p profile area of the SPF ($\mathrm{m^2}$)
A fin cross-sectional area ($\mathrm{m^2}$)
$A_p, \widehat{A}_p$ dimensionless total and reduced profile area of the SPF, $A_p = a_p(h_b/k)^2$; $\widehat{A}_p = A_p/2$
Bi, Bi_l, Bi_a, Bi_v dimensionless Biot numbers based on the thickness (radius), height, profile area (volume) of the SPF(CPF), respectively
E_f extension factor of the total fin heat transfer surface, F/A
F area of the total fin heat transfer surface ($\mathrm{m^2}$)
g_b fin base thermal conductance, Q_b/ϑ_b ($\mathrm{W\,K^{-1}}$)
g_l thermal conductance of a fin with insulated lateral surfaces, kA/l ($\mathrm{W\,K^{-1}}$)
g_p thermal conductance of the fin prime surface with an area equal to A ($\mathrm{W\,K^{-1}}$)
G dimensionless thermal conductance of the SPF and CPF, $G = \widehat{G}_z/(Bi_a^2/2)^{1/3}$; $G = \widehat{G}_c/[Bi_v^3/(4\pi)]^{3/5}$
G_b dimensionless fin thermal conductance, g_b/g_l
$G_c, \widehat{G}_c$ dimensionless total and reduced thermal conductance of the CPF, $G_c = g_b(h_b/k^2)$, $\widehat{G}_c = G_c/(4\pi)$
G_{cV} dimensionless thermal conductance of the CPF per unit volume, G_c/V
G_d dimensionless relative thermal conductance of a fin, $G_b/G_{b,n=0}$
$G_z, \widehat{G}_z$ dimensionless total and reduced thermal conductance of the SPF, $G_z = g_b/(zk)$; $\widehat{G}_z = G_z/2$
G_{zA_p} dimensionless specific thermal conductance of the SPF, G_z/A_p
h, h_e heat transfer coefficients on the fin lateral and tip surfaces ($\mathrm{W\,m^{-2}\,K^{-1}}$)
K fin augmentation factor (effectiveness), g_b/g_p
k thermal conductivity of the fin material ($\mathrm{W m^{-1}\,K^{-1}}$)
l fin height (m)
n given exponent in the heat transfer Eq. (1)
N dimensionless thermo-geometrical fin parameter, $l\sqrt{h_b P/(kA)}$
P perimeter of the fin cross section (m)
r radius of the CPF (m)
Q_b heat flow dissipated by the fin (W)
t fin temperature (K)
t_a environment (ambient) temperature (K)
T dimensionless fin temperature, ϑ/ϑ_b
T_e dimensionless fin tip temperature, ϑ_e/ϑ_b
v volume of the CPF ($\mathrm{m^3}$)
$V, \widehat{V}$ dimensionless total and reduced volume of the CPF, $V = v(h_b/k)^3$, $\widehat{V} = V/(4\pi)$
x space coordinate (m)
X dimensionless space coordinate, x/l
z width of the SPF (m)

Greek symbols
δ fin thickness (m)
μ preexponential factor in Eqs. (8), (11) and below
ϑ local temperature difference between a fin and environment, $t - t_a$ (K)
φ relative percent discrepancy between the closed-form and recurrent solutions for T_e and T (%)
ψ fin aspect ratio (fin height to half thickness or half radius ratio), Bi_l/Bi
$\omega = h_e/h_{x=0}$ ratio of heat transfer coefficients on the fin tip and lateral surfaces, a_e/a

Subscripts and superscripts
a ambient medium (environment)
b fin base (at $X = 1$)
e fin tip (at $X = 0$)
* fin with an insulated tip

whole surface of the fin. For a limited number of n the analytical solutions can be expressed via special functions. However, all these formulae are implicit with respect to the dimensionless temperature T_e of the fin tip and their analytic inversion into an explicit form is possible only for values of n indicated above. The effect of the boundary condition at a fin tip on the performance of the fin was investigated in [13]. Sen and Trinh [14], Yeh and Liaw [15] as well as Liaw and Yeh [16,17] showed that the exact inverse solution exists in the form of the three-parameter hypergeometric function for the arbitrary values of n. This representation is also an implicit expression with respect to T_e.

An analysis of the longitudinal and annular fins and spines is performed by Laor and Kalman in [18]. The effect of the temperature dependence of the heat transfer coefficient on the fin performance and optimum dimensions was analyzed for the fins subjected to various heat transfer modes, specifically, free convection ($n = 1/4$ and $n = 1/3$), nucleate boiling ($n = 2$) and radiation ($n = 3$). The governing equations were solved numerically by the trial-and-error method. The efficiency and optimum fin performance as well as dimensions were described graphically and by simple correlations for the above-mentioned cases.

Performance of single fins of different types, specifically straight plate, annular and pin fins, with different forms of profile area, in particular, uniform (of constant thickness or diameter), convex and concave parabolic, and triangular has been investigated in the papers by Mokheimer [19,20] for the natural convection heat transfer mode. Results have been presented in the plots of fin efficiency *vs* thermo-geometrical parameter of a fin. But the latter parameter involves the heat transfer coefficient calculated locally and averaged along the fin. Since such coefficient is not given by the problem input data it needs to be related with the specified data. Otherwise the presented plots for the fin efficiency is impossible to use in practice.

An interesting method for analysis of individual longitudinal fins with rectangular profile, uniform heat transfer coefficient, and a non-insulated tip has been proposed by Razelos and Krikkis [21]. It has been shown that, with a slight modification of the dimensionless parameters that describe this problem, the analysis of the fin with a non-insulated tip becomes similar to those for the fin with an insulated tip. It was done by introduction of complex variable $\omega\sqrt{Bi}$ to account for the non-insulated tip, where $\omega = h_e/h$ is ratio of the heat transfer coefficients on the tip surface and on the lateral surfaces of a fin. We will use this idea for analysis of single fins with a non-insulated tip and with a heat transfer coefficient depending on the local temperature excess.

The detailed examination of the papers related to the analysis and optimization of a single fin subject to the power dependence of h on ϑ allows us to make a conclusion that the obtained solutions are usually presented in a graphical form for certain heat transfer modes, i.e. for the definite values of n. To our knowledge, the closed-form formulae to determine directly the fin tip temperature excess T_e and the heat flow dissipated by the fin under the continuous variation of the given independent variables n, N and $\omega^2 Bi$ were absent in the available literature. The expressions of such kind were first obtained in [1,2]. The closed-form inverse solution of the 1-D heat conduction problem for a single fin with a constant cross section and an insulated tip ($\omega^2 Bi = 0$) was derived in [1]. It was obtained in a form $N/N_0 = T_e^{-\mu n}$ where N_0 is a well-known expression for N at $n = 0$ that corresponds to the uniform heat transfer coefficient over the whole surface of the fin. In [1], a coefficient μ was found to be equal to 0.4 after the fitting procedure with the results of the numerical integration in the range $0.1 \leqslant T_e \leqslant 1$ and $-7 \leqslant n \leqslant 7$. The recurrent direct solution with the high convergence rate to calculate accurately T_e for given values of n and N is obtained in [2] by the inversion of the closed-form inverse solution. The high convergence rate was achieved by using the linearization procedure. The value of T_e allowed one to compute the fin base thermal conductance and augmentation factor.

Hence, taking into consideration the aforementioned remarks, the present paper has the following objectives:

- The results of [1,2] have to be generalized to include fins with a non-insulated tip, i.e. the effect of the fin tip heat loss must be taken into account.
- The closed-form inverse solution has to be derived for straight fins of uniform cross section with a non-insulated tip.
- The obtained extended inverse solution has to be inverted into the direct recurrent solution to determine T_e and temperature distribution along the fin height.
- The linearization method developed in our paper [2] has to be used to transform the direct recurrent formulae for T_e and for the temperature distribution along the fin height with poor rate of convergence into the corresponding expressions with a very high convergence rate.
- The formulae, which allow one to determine the arbitrary geometrical or thermal parameter of the SPF and CPF using the given value of any other of its geometrical or thermal parameters and specified values of n, N or (and) G, have to be systematized and tabulated.
- All these contributions must allow to obtain explicit closed-form expressions for optimum geometrical and thermal characteristics of the SPF and CPF in the second part of this study.

The considered problem will be solved using the following set of the extended Murray–Gardner assumptions [5]:

(1) The thermal process in the fin and overall heat transfer problem are steady state.
(2) The fin material is homogeneous and isotropic.
(3) The local heat transfer coefficient on the lateral faces and tip of the fin is a power function of the local temperature difference between the fin surface and environment with the same values of exponents n and, in general case, different values of coefficients a for the lateral faces and a_e for the tip, i.e. with different values of fin tip ratio $\omega = h_e/h_{x=0} = a_e/a$.
(4) The temperature of the environment is uniform and constant.
(5) The fin thickness is small compared with its height and length, so that temperature gradients across the fin thickness and heat transfer from the edges of the fin may be neglected (1-D heat conduction assumption).
(6) The temperature of the fin base is uniform and constant.
(7) There is a perfect thermal contact between the fin and the prime surface at the base.
(8) There are no heat sources within the fin itself.

2. Extension of the [1] solution to account for a fin tip heat loss

Earlier [1], we have solved a heat conduction problem for a straight fin of rectangular profile (a longitudinal fin or a spine) with a constant cross-sectional area A of an arbitrary form having the perimeter P. The local heat transfer coefficient along the fin height was considered to exhibit the power-law type dependence on the local temperature difference between the fin surface and the environment, i.e.

$$h = a(t - t_a)^n = a\vartheta^n. \quad (1)$$

It is assumed that the heat transfer coefficient for the tip surface of a fin as function of the tip temperature excess $t_e - t_a$ has the same form as Eq. (1) with the same value of exponent n but, in general case, with different coefficient $a_e \neq a$. Thus, the dimensionless ratio of $\omega = h_e/h_{x=0} = a_e/a$ was introduced into analysis to account for the heat loss from the fin tip. In general case, value of $\omega \geqslant 0$. For fins with an insulated tip $\omega = 0$.

The one-dimensional equation of the steady-state heat conduction for a fin is analyzed. This equation in terms of dimensionless variables $X = x/l$ and $T = (t - t_a)/(t_b - t_a) = \vartheta/\vartheta_b$ can be written as follows:

$$\frac{d^2T}{dX^2} - N^2T^{n+1} = 0, \quad (2)$$

where the thermo-geometrical parameter N of a fin in Eq. (2) is defined as

$$N = l\sqrt{\frac{h_bP}{kA}} = l\sqrt{\frac{a\vartheta_b^nP}{kA}}. \quad (3)$$

The parameters h_b, l, A, and P in Eq. (3) represent the heat transfer coefficient on the lateral surfaces at the fin base, fin height, area and perimeter of the fin cross section, respectively, k is the thermal conductivity of the fin material. Note that the height coordinate X has its origin at the fin tip and has a positive orientation from the fin tip to the fin base.

The first boundary condition to Eq. (2) to account for the fin tip heat transfer can be written in dimensionless form as

$$dT/dX|_{X=0} = -\omega Bi_l T_e^{n+1} = -\omega\sqrt{Bi}NT_e^{n+1}, \quad X = 0, \quad (4)$$

where $\omega = h_e/h = a_e/a$ is a ratio of heat transfer coefficients from the fin tip surface and its lateral surfaces at the tip coordinate ($X = 0$), $Bi_l = h_bl/k$ is the Biot number based on the fin height l and on the fin base heat transfer coefficient h_b, $Bi = h_bA/(kP)$ is the transverse Biot number of a fin.

The second boundary condition at the fin base is the same as that for the fin with an insulated tip:

$$T = 1, \quad X = 1. \quad (5)$$

Taking into account the above boundary conditions, we get the following solution of Eq. (2) for the thermo-geometrical parameter N of the fin with a non-insulated tip in form of definite integral likewise to definite integral in [1] for a fin with an insulated tip,

for $n \neq -2$

$$N = \int_{T_e}^{1} dT\Big/\sqrt{[2/(n+2)]\{T^{n+2} - T_e^{n+2}[1 - (n+2)\omega^2BiT_e^n/2]\}}, \quad (6)$$

Using the derivative of the general solution of Eq. (2) at the fin base, we get the following expression for the thermal conductance of a fin with a non-insulated tip,

for $n \neq -2$

$$G_b = \frac{g_bl}{kA} = N\sqrt{[2/(n+2)]\{1 - T_e^{n+2}[1 - (n+2)\omega^2BiT_e^n/2]\}}. \quad (7)$$

The latter equation involves the same expression in square brackets as the corresponding brackets in the denominator of Eq. (6). If the tip of a fin is insulated ($\omega^2Bi = 0$), then the value in square brackets of Eqs. (6) and (7) is equal to 1 and these equations are simplified to corresponding expressions in [1]. In this case functions N *vs* T_e and G_b *vs* T_e depend on single parameter n only, whereas two other parameters ω and Bi are added for the fin with a non-insulated tip. Generally, parameter ω is always positive and can be lesser, equal to, or greater than 1. Parameter Bi is the dimensionless expression for transverse dimension (half-thickness or half-radius) of a fin. In the following analysis we take into account all three parameters n, ω and Bi for the fins with a non-insulated tip. Notice that parameters ω and Bi will occur in all following expressions in the form of product ω^2Bi or $\omega\sqrt{Bi}$ only and can be treated as a single parameter.

Numerical computations of the function N *vs* T_e are performed in the present study using Eq. (7) in the range of independent variable $0 < T_e \leqslant 1$ for a single given value of parameter $\omega = 1$ ($a_e = a$) and variation of parameters n and Bi in the range $-0.5 \leqslant n \leqslant 5$ and $0 \leqslant Bi \leqslant 0.015$, respectively. The results are processed by the least square method and presented in the same form as in [1], i.e.

$$N/N_0 = T_e^{-\mu n}, \quad (8)$$

where N_0 denotes the N value for $n = 0$. Function N_0 *vs* T_e with account for the fin tip heat loss (ω^2Bi) can be written as

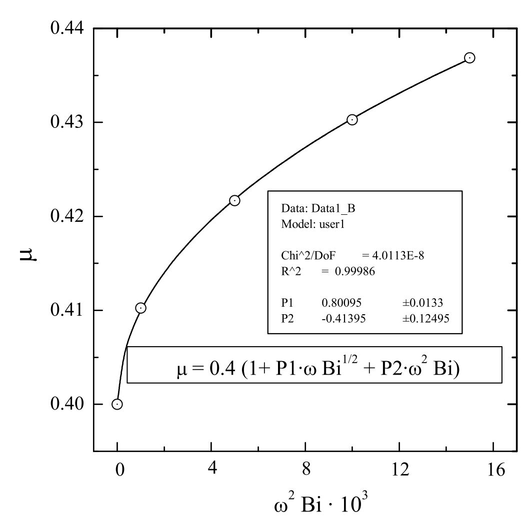

Fig. 1. Preexponential factor μ in Eq. (8) plotted as a function of the product ω^2Bi for a straight fin with a non-insulated and, particularly, an insulated tip predicted by Eqs. (6), (8), and (10) (dot-centered open circles) and an approximation curve for these data by the second-order polynomial with respect to $\omega\sqrt{Bi}$ (solid line).

$$N_0 = \mathrm{arcosh}\left(\frac{1}{T_e\sqrt{1-\omega^2 Bi}}\right) - \mathrm{arcosh}\left(\frac{1}{\sqrt{1-\omega^2 Bi}}\right). \quad (9)$$

This equation may be rewritten also in the following equivalent form,

$$N_0 = \ln\{[1+\sqrt{1-T_e^2(1-\omega^2 Bi)}]/[T_e(1+\omega\sqrt{Bi})]\}. \quad (10)$$

The preexponential factor μ in Eq. (8) is not a constant value being equal to 0.4, as was determined in [1] for the fin with an insulated tip ($\omega = 0$), but it depends only on the complex parameter $\omega^2 Bi$ defined above. The plot μ *vs* $\omega^2 Bi$ calculated using Eqs. (8) and (10) is displayed in Fig. 1 by dot-centered open circles for $\omega^2 Bi = 0$, 0.001, 0.005, 0.01, and 0.015. The approximation of these data by the second-order polynomial with respect to $\omega\sqrt{Bi}$ is presented in Fig. 1 by a solid line:

$$\mu = 0.4(1 + 0.8\omega\sqrt{Bi} - 0.414\omega^2 Bi). \quad (11)$$

It is seen that approximation curve practically coincides with data of numerical computation (dot-centered open circles). Thus, approximate formula is of a high accuracy for the values of $\omega^2 Bi$ in the range $0 \leqslant \omega^2 Bi \leqslant 0.015$. Out of this range Eqs. (11) and (8) are less accurate.

3. Recurrent direct solution

The inversion of the closed-form inverse solution Eq. (8) in combination with Eq. (9) allows to find the recurrent direct solution for the straight fin of a constant cross section. The following denotation is introduced for convenience of the implementation of this procedure,

$$Z = NT_e^{\mu n} + \mathrm{arcosh}\left(\frac{1}{\sqrt{1-\omega^2 Bi}}\right) = NT_e^{\mu n} + \ln\frac{1+\omega\sqrt{Bi}}{\sqrt{1-\omega^2 Bi}}. \quad (12)$$

The next relation can be obtained using Eqs. (8), (9), and denotation expressed by Eq. (12),

$$\mathrm{arcosh}\left(\frac{1}{T_e\sqrt{1-\omega^2 Bi}}\right) = Z, \quad (13)$$

from which the implicit recurrent formula to determine T_e follows:

$$T_e = 1/\left(\sqrt{1-\omega^2 Bi}\cosh Z\right). \quad (14)$$

Using Eq. (12) and formulae of the hyperbolic trigonometry, Eq. (14) may be expressed also in the equivalent form,

$$T_e = 1/[\cosh(NT_e^{\mu n}) + \omega\sqrt{Bi}\sinh(NT_e^{\mu n})]. \quad (15)$$

If heat transfer coefficient over the whole fin surface is uniform ($n = 0$), so the recurrent equation (15) is transformed into the well-known explicit expression,

$$T_e = 1/(\cosh N + \omega\sqrt{Bi}\sinh N). \quad (16)$$

Unfortunately, the recurrent direct solution equation (14) or (15) has poor convergence. Therefore, it will be transformed into the expression with the very high convergence rate by the same linearization method, that we used in [2] for the fins with an insulated tip. Denote the RHS of Eq. (14) as $F(Z)$ and increment of T_e by the symbol Δ. Then, the following expression is obtained using the indirect differentiation rule,

$$T_e + \Delta = \frac{1}{\sqrt{1-\omega^2 Bi}\cosh Z} + \frac{dF}{dZ}\frac{dZ}{dT_e}\Delta. \quad (17)$$

From Eqs. (14) and (12) follows, respectively, that

$$\frac{dF}{dZ} = -\frac{\sinh Z}{\sqrt{1-\omega^2 Bi}\cosh^2 Z} = -\frac{\tanh Z}{\sqrt{1-\omega^2 Bi}\cosh Z}, \quad (18)$$

$$\frac{dZ}{dT_e} = \frac{\mu n N T_e^{\mu n}}{T_e} = \frac{A_Z}{T_e}, \quad (19)$$

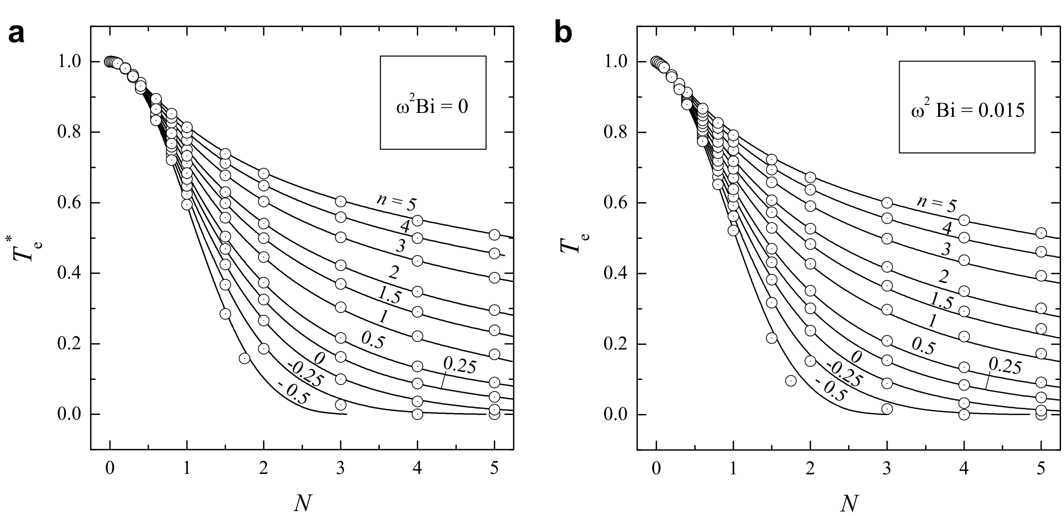

Fig. 2. Dimensionless temperature of the fin tip T_e^* and T_e plotted as a function of thermo-geometrical parameter N for a straight fin with an insulated tip ($\omega^2 Bi = 0$, a) and non-insulated tip ($\omega^2 Bi = 0.015$, b) predicted by numerical evaluation of the integral in Eq. (6) (solid lines) and by means of recurrent equation (23) for different values of exponent n (dot-centered open circles).

where it is denoted that

$$A_Z = \mu n N T_e^{\mu n} \tanh Z. \tag{20}$$

To get the expression determining an increment Δ one has to substitute Eqs. (18) and (19) into Eq. (17),

$$T_e + \Delta = \frac{1}{\sqrt{1-\omega^2 Bi}\cosh Z} - \frac{A_Z}{T_e\sqrt{1-\omega^2 Bi}\cosh Z}\Delta. \tag{21}$$

As it follows from Eq. (21),

$$\Delta = \frac{1 - T_e\sqrt{1-\omega^2 Bi}\cosh Z}{\sqrt{1-\omega^2 Bi}\cosh Z + (A_Z/T_e)}. \tag{22}$$

Upon substitution of this expression into the RHS of Eq. (21) and taking into account that in an incremental form $T_{e,(j+1)} = T_{e,j} + \Delta$ with subscripts (j) and ($j+1$) denoting the iteration number, we get the following recurrent formula to determine T_e for given values of n, N and $\omega^2 Bi$,

$$T_e = \frac{1 + A_Z}{\sqrt{1-\omega^2 Bi}\cosh Z + (A_Z/T_e)}. \tag{23}$$

The subscripts (j) and ($j+1$) in Eq. (23) are omitted for simplicity. An arbitrary value of T_e in the range $0 < T_e \leqslant 1$ can be taken as the zero approximation in the RHS of Eq. (23). The convergence rate of Eq. (23) is so high. It is enough only 1–3 iterations to obtain a relative difference between two successive approximations better than 10^{-4} % in the range of $0 < N \leqslant 5$; $-0.5 \leqslant n \leqslant 5$ for $0 \leqslant \omega^2 Bi \leqslant 0.015$. The comparison of the results obtained by means of direct recurrent equation (23) and the inverse closed-form equation (8) shows that they are in excellent agreement with each other and with the numerical evaluation of integral in Eq. (6) in the whole range of the parameters N, n and $\omega^2 Bi$. This is clearly seen in Fig. 2a and b for the fins with an insulated and non-insulated tip, respectively. A maximum discrepancy between the former and latter cases not exceed 2% for positive $0 \leqslant n \leqslant 5$ at $0 \leqslant \omega^2 Bi \leqslant 0.015$ and $0 \leqslant N \leqslant 5$, whereas it somewhat increases for negative $-0.5 \leqslant n < 0$. The difference in T_e *vs* N and n for the fins with insulated ($\omega^2 Bi = 0$) and non-insulated tip ($\omega^2 Bi = 0.015$) shown in Fig. 3 is relatively small. It is most pronounced for the low values of N. The corresponding curves T_e *vs* N for so called "asymptotical" fins with $N \geqslant 3.5$ practically merge.

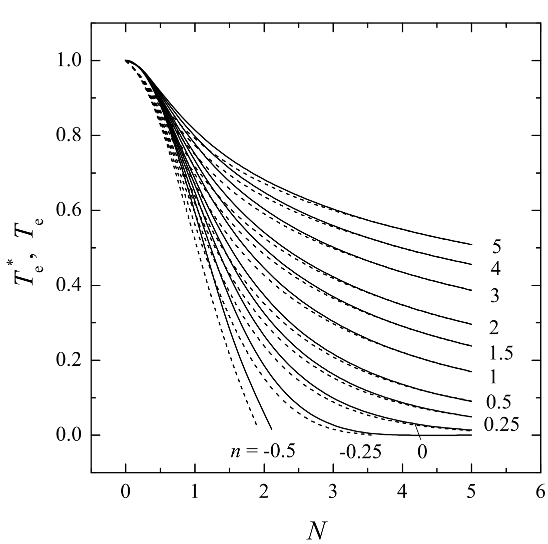

Fig. 3. Comparison between the curves T_e^* *vs* N (solid lines) and T_e *vs* N (short-dashed lines) for straight plate fins with an insulated tip ($\omega^2 Bi = 0$) and non-insulated tip ($\omega^2 Bi = 0.015$) predicted by means of recurrent equation (23) for different values of exponent n.

4. Closed-form direct solution

4.1. Dimensionless temperature of the fin tip

The following approach based on the high convergence rate of Eq. (23) is proposed in this section to obtain the closed-form direct solution for the accurate determination of the relationship between the dimensionless temperature of the fin tip T_e and parameters n, N and $\omega^2 Bi$.

First, the relationship between $T^*_{e,app}$ and parameters N and n is determined for a fin with an insulated tip ($\omega^2 Bi = 0$) using the data of numerical calculations, approximated by the following function of N:

$$T^*_{e,app} = \cosh(P_1 N + P_2 N^{1/2}) / \cosh(N + P_3 N^3). \tag{24}$$

The coefficients P_1–P_3 are expressed by the following functions of n.

For $0 \leqslant n \leqslant 5$

$$\begin{aligned} P_1 &= 3.7n/(1 + 3.55n), \\ P_2 &= -2.24n/(1 + 4.62n + 0.62n^2), \\ P_3 &= 0. \end{aligned} \tag{25}$$

If the heat transfer coefficient over the whole fin surface is uniform, i.e. $n = 0$, then all coefficients P_1–P_3 in Eq. (24) according to Eq. (25) are equal to zero. In this case Eq. (24) converts into the well-known exact solution Eq. (16) for a straight fin or spine of constant cross section with an insulated tip ($\omega\sqrt{Bi} = 0$).

For $-0.75 \leqslant n < 0$

$$\begin{aligned} P_1 &= -1.993n/(1 - 0.9n - 1.563n^2), \\ P_2 &= 1.88n/(1 - 1.115n - 0.87n^2), \\ P_3 &= -0.121n/(1 + 1.546n + 0.581n^2). \end{aligned} \tag{26}$$

The examples of approximations using the relationship T_e^* *vs* N according to Eq. (24) for three values of n, specifically, $n = -0.5$, 0 and 5 are presented in Fig. 4a–c, respectively. The n dependence of the coefficients P_1 and P_2 in Eq. (24) for $n \geqslant 0$ expressed by Eq. (25) is shown in Fig. 5a and b.

Second, the obtained value of $T^*_{e,app}$ is substituted into the RHS of Eqs. (12), (20), and (23) instead of T_e. As a consequence, the LHS of Eq. (23) gives the final refined and practically exact value of T_e. This result is confirmed by Fig. 6 for the fins with insulated ($\omega^2 Bi = 0$, a) and

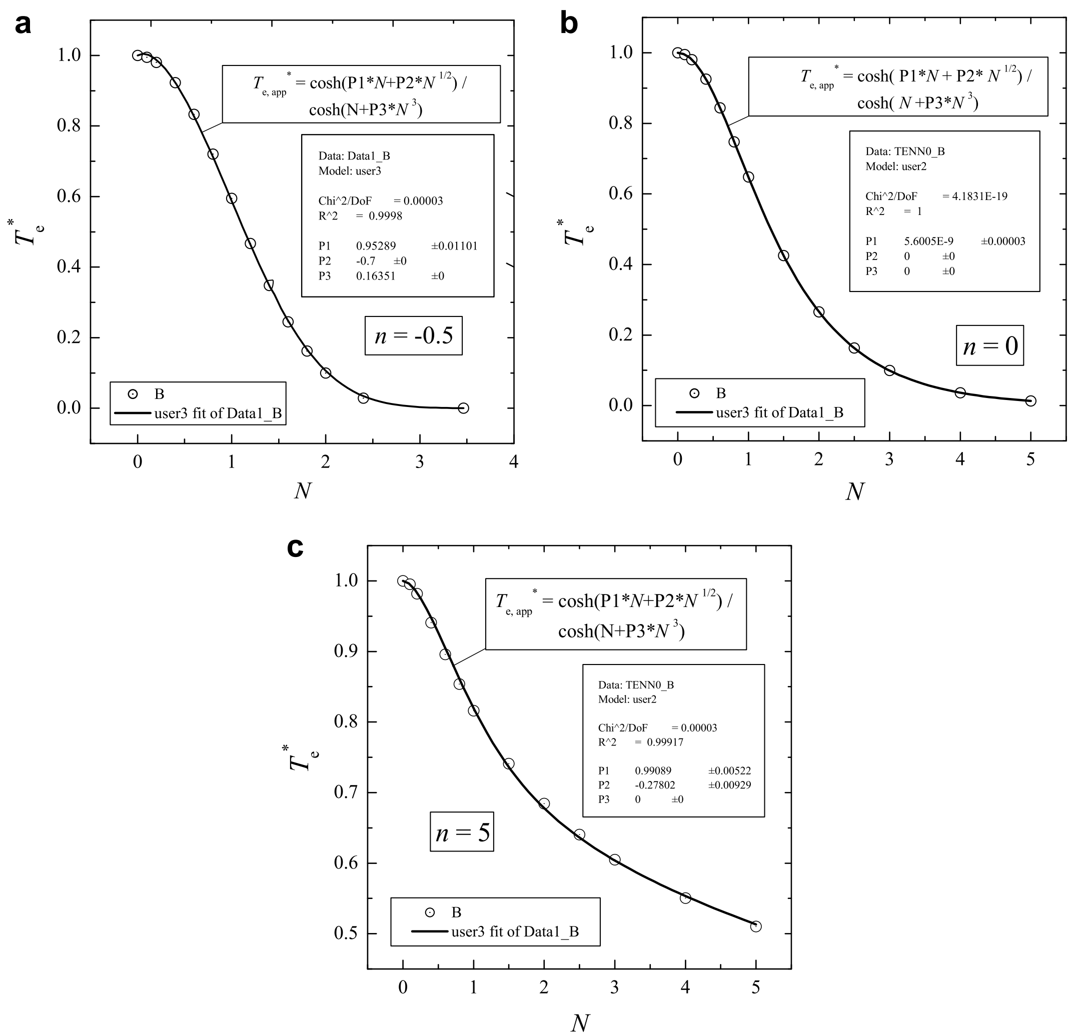

Fig. 4. Dimensionless temperature of the fin tip $T^*_{e,app}$ plotted as a function of thermo-geometrical parameter N for a straight fin with an insulated tip obtained by numerical evaluation of the integral in Eq. (6) (dot-centered open circles) and its fit (solid curves) by means of Eq. (24) for $n = -0.5$ (a), $n = 0$ (b), and $n = 5$ (c).

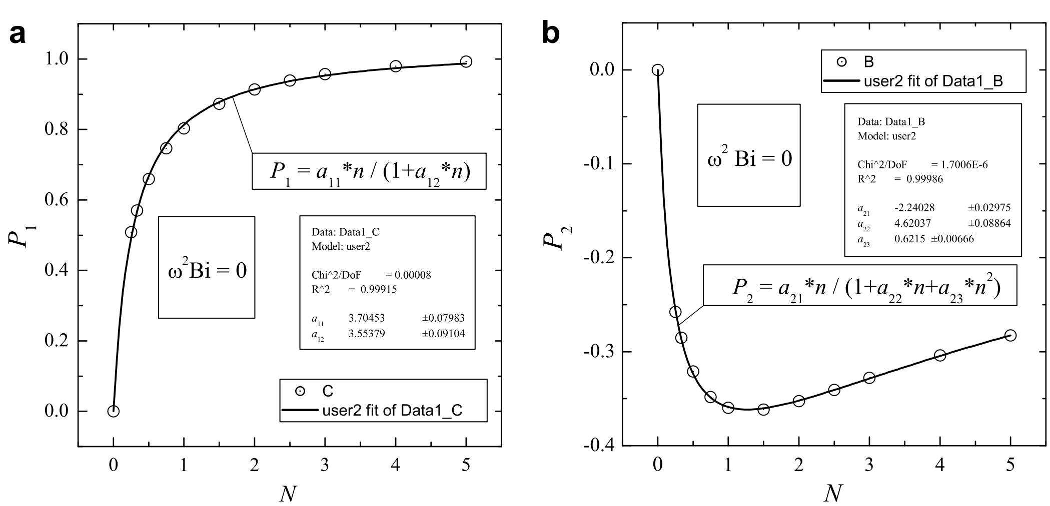

Fig. 5. Coefficients P_1 and P_2 in Eq. (24) plotted as functions of exponent n (a and b, respectively) shown by dot-centered open circles and its fit by homographic functions equation (26) at $n > 0$ (solid lines) for a straight fins with an insulated tip ($\omega^2 Bi = 0$).

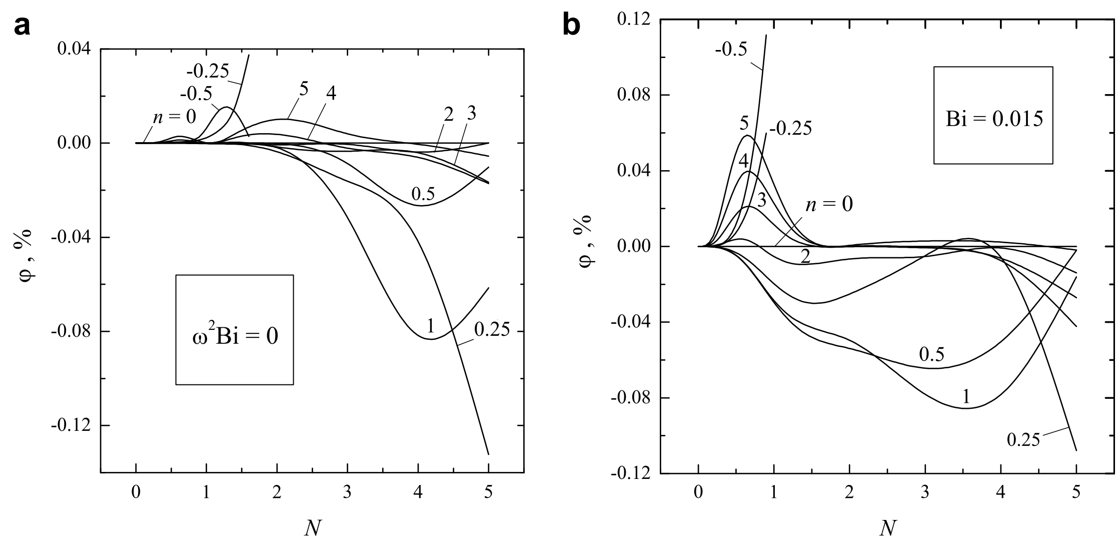

Fig. 6. Percent relative discrepancy φ between the values of T_e predicted using closed-form equations (23)–(26) and recurrent equation (23) for straight fins with an insulated tip ($\omega^2 Bi = 0$, a) and non-insulated tip ($\omega^2 Bi = 0.015$, b).

non-insulated tips ($\omega^2 Bi = 0.015$, b), where the relative discrepancy φ between T_e calculated using closed-form Eqs. (23)–(26) and recurrent equation (23) is plotted against the fin parameter N. Now, it could be seen that the maximum relative discrepancy is $-0.14\% \leqslant \varphi \leqslant 0.04\%$ for $\omega^2 Bi = 0$, $0 < N \leqslant 5$, and $-0.5 \leqslant n \leqslant 5$. Analogous comparisons for $\omega^2 Bi = 0.015$ in the same range of independent variables give the maximum relative discrepancy within the range $-0.12\% \leqslant \varphi \leqslant 0.12\%$. Thus, Eq. (23) in combination with Eqs. (24)–(26) can be considered as a direct closed-form high accuracy solution to determine the relationship between T_e and parameters N, n and $\omega^2 Bi$. The results of these evaluations are practically identical to those calculated by recurrent formula (23) and plotted in Fig. 3.

4.2. Temperature distribution in a fin

The expression for the product of the dimensionless coordinate X along a fin height and the thermo-geometrical parameter N of the fin with a non-insulated tip is the same definite integral as in Eq. (6), but with the other limits of integration (from T_e to T instead of from T_e to 1), for $n \neq -2$

$$X \cdot N = \int_{T_e}^{T} \mathrm{d}T \Big/ \sqrt{[2/(n+2)]\{T^{n+2} - T_e^{n+2}[1-(n+2)\omega^2 Bi T_e^n/2]\}}, \tag{27}$$

Obtain a general expression for the evaluation of the considered definite integral with an arbitrary upper limit. To do this, it is suitable to denote

$$\kappa = [1-(n+2)\omega^2 Bi T_e^n/2]^{1/(n+2)}, \tag{28}$$

and introduce another variable of integration $\xi = T_e/T$. Using these definitions, Eq. (6) can be readily transformed into the form, for $n \neq -2$

$$N = \sqrt{\frac{n+2}{2(T_e\kappa)^n}} \int_{T_e\kappa}^{\kappa} \frac{\mathrm{d}\xi}{\sqrt{\xi^{2-n} - \xi^4}}, \tag{29}$$

According to Eqs. (8) and (9), the parameter of the fin is equal to

$$N = \operatorname{arcosh}(1/T_e\sqrt{1-\omega^2 Bi}) - \operatorname{arcosh}(1/\sqrt{1-\omega^2 Bi})/T_e^{\mu n}. \tag{30}$$

By equating these expressions it is obtained

$$I(T_e, n, \omega^2 Bi) = \int_{T_e\kappa}^{\kappa} \frac{\mathrm{d}\xi}{\sqrt{\xi^{2-n} - \xi^4}} = \sqrt{\frac{2(T_e\kappa)^n}{(n+2)}} \times \left\{ \frac{\operatorname{arcosh}[1/(T_e\sqrt{1-\omega^2 Bi})] - \operatorname{arcosh}(1/\sqrt{1-\omega^2 Bi})}{T_e^{\mu n}} \right\}. \tag{31}$$

The integral in Eq. (27) will transformed in a similar manner into the following form:

for $n \neq -2$

$$X \cdot N = \sqrt{\frac{n+2}{2(T_e\kappa)^n}} \int_{\frac{T_e}{T}\kappa}^{\kappa} \frac{\mathrm{d}\xi}{\sqrt{\xi^{2-n} - \xi^4}}. \tag{32}$$

It can be seen that the integral in RHS of Eq. (32) is nearly equivalent to that in the RHS of Eq. (29), but with another bottom limit of integration $T_e\kappa/T$ instead of $T_e\kappa$. In order to obtain the full equivalency, it is sufficient to replace the value $\omega^2 Bi$ in the RHS of Eq. (31) by $\omega^2 Bi T^n$. Therefore, if the bottom limit in the integral of Eq. (31) is denoted by $(T_e/T)\kappa$, the following expression is obtained:

$$I(T_e/T, n, \omega^2 Bi) = \int_{\frac{T_e}{T}\kappa}^{\kappa} \frac{d\xi}{\sqrt{\xi^{2-n} - \xi^4}} = \sqrt{\frac{2(T_e\kappa)^n}{(n+2)}} \times \left\{ \frac{\text{arcosh}\{1/[(T_e/T)\sqrt{1-\omega^2 BiT^n}]\} - \text{arcosh}(1/\sqrt{1-\omega^2 BiT^n})}{(T_e/T)^{\mu n}} \right\}. \quad (33)$$

Upon substitution of this expression into Eq. (32), the inverse formula for the temperature distribution in a fin follows:

$$X = \frac{\text{arcosh}\left\{1/[(T_e/T)\sqrt{1-\omega^2 BiT^n}]\right\} - \text{arcosh}(1/\sqrt{1-\omega^2 BiT^n})}{NT_e^{\mu n} T^{(\frac{1}{2}-\mu)n}}. \quad (34)$$

Eq. (34) can be readily transformed into the recurrent direct expression to determine T for given X, n, N and $\omega^2 Bi$ with the T_e obtained using Eqs. (23)–(26). Denote

$$U = \sqrt{1 - \omega^2 BiT^n}, \quad (35)$$

$$V = NXT_e^{\mu n} T^{(\frac{1}{2}-\mu)n}, \quad (36)$$

$$W = V + \text{arcosh}(1/U). \quad (37)$$

Using Eq. (34) and the denotations above, one get

$$\text{arcosh}\{1/[(T_e/T)U]\} = W. \quad (38)$$

Then, the following recurrent expression to determine T for given values of X, n, N and $\omega^2 Bi$ is obtained

$$T = T_e U \cosh W. \quad (39)$$

Unfortunately, Eq. (39) has a poor convergence rate. Therefore, this equation is transformed into the recurrent formula with a high convergence rate using the same linearization approach, as in the development of the recurrent equation (23). For simplicity's sake the following denotations are introduced in addition to the ones above in Eqs. (35)–(37),

$$U_1 = n(1 - U^2)/(2U), \quad (40)$$

$$U_2 = n\sqrt{1 - U^2}/(2U^2), \quad (41)$$

$$R = Un\left[\left(\frac{1}{2} - \mu\right)V - U_2\right]. \quad (42)$$

As a result, to determine the temperature profile along a fin with a non-insulated tip the following direct equation is obtained:

$$T = T_e \frac{(U + U_1)\cosh W - R\sinh W}{1 + T_e(U_1\cosh W - R\sinh W)/T}. \quad (43)$$

One can easily see from Eqs. (35)–(37) and (40)–(42) that for fins with an insulated tip $\omega^2 BiT_e^n = 0$ and according to Eqs. (35), (40) and (41) $U = 1$, $U_1 = 0$, and $U_2 = 0$. From Eqs. (36), (37) and (42) $V = NXT_e^{0.4n}T^{0.1n}$, $W = V$ and $R = 0.1nV$. Upon substitution of these values into Eq. (43), one obtains

$$T^* = T_e^* \frac{\cosh V - R\sinh V}{1 - T_e^* R\sinh V/T^*}. \quad (44)$$

The linear dependence

$$T^* = T_e^* + X(1 - T_e^*). \quad (45)$$

can be taken as the zero approximation for T^* in the RHS of Eq. (44),

The numerical calculations of the temperature profiles for the fins with different values of n, N and $\omega^2 Bi$ showed that even a single calculation using Eq. (44) with a linear approximation by Eq. (45) is sufficient to determine the value of T^* in the LHS of Eq. (44) with the relative error less than 0.01%. Two–three iterations allow one to determine the value of T^* with the relative error 10^{-3}–10^{-4}%. Thus, Eqs. (44) and (45) can be assumed as a closed-form explicit equation for the temperature profile in a fin with an insulated tip.

The near exact value of T can be determined in the LHS of Eq. (43) for the fin with a non-insulated tip if the value of T^* calculated using Eqs. (44) and (45) is used as T in the RHS of Eq. (43). Therefore, Eq. (43) with denotations expressed by Eqs. (35)–(37), and (40)–(42) can be considered as a high accuracy closed-form solution for the temperature profile in a fin with a non-insulated and, in particular, insulated tip. The temperature distribution T *vs* X in a straight fin with an insulated ($\omega^2 Bi = 0$, solid lines) or non-insulated tip ($\omega^2 Bi = 0.015$, short-dashed lines) calculated using these equations is shown in Fig. 7 for $N = 1$ and different n. Apparently, for $\omega^2 Bi = 0$ the gradient of T at the tip of fin where $X = 0$ is equal to zero as the boundary condition for an insulated tip requires. All curves for $\omega^2 Bi = 0.015$ and different n pass steeper than corresponding curves for $\omega^2 Bi = 0$ and gradient of T at $X = 0$ is not equal to zero (because the fin tip is non-insulated).

The comparison between the temperature distribution in a cylindrical copper rod where heat transfer by film boiling with water occurs is shown in Fig. 8. The calculations

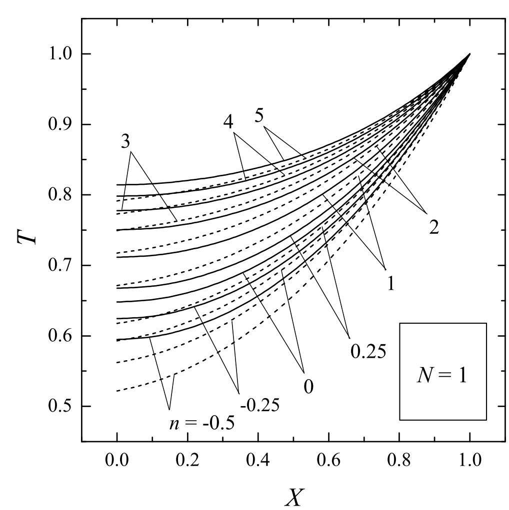

Fig. 7. Temperature distribution in straight fins with an insulated tip ($\omega^2 Bi = 0$, solid lines) and non-insulated tip ($\omega^2 Bi = 0.015$, short-dashed lines) predicted by Eqs. (43)–(45) for given $N = 1$ and different values of exponent n.

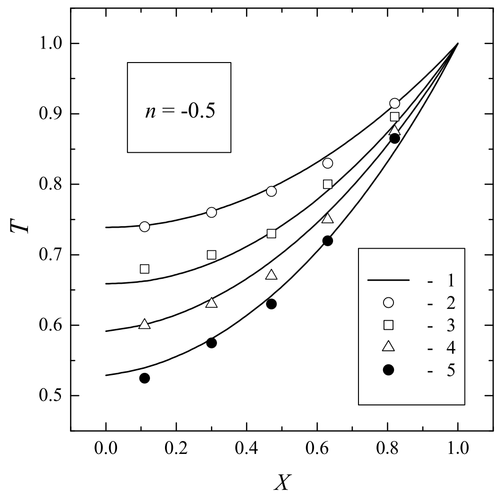

Fig. 8. Temperature distribution in the cylindrical copper rod with an insulated tip (upper two curves) and non-insulated tip (lower two curves) predicted by Eqs. (43)–(45) (solid lines 1), and experimental data of [16]: points 2–5 present data in film boiling with water ($n = -0.5$) on the rod of $d = 25$ mm, $l = 85$ mm, and $\psi = 13.6$ at $\vartheta_b = 254$, 140, 90 and 84 °C ($N = 0.77$, 0.90, 0.94 and 1.03, respectively).

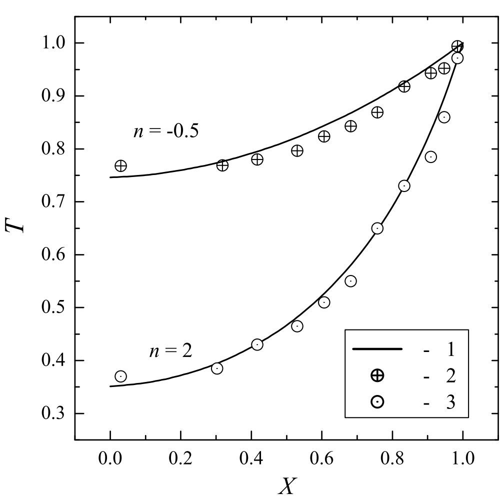

Fig. 9. Temperature distribution in the cylindrical copper rod with a non-insulated tip predicted by Eqs. (43)–(45) (solid lines) and experimental data of [22]: points 2 and 3 present data in film ($n = -0.5$) and nucleate pool boiling ($n = 2$) with Freon-113 on the rod of $d = 14$ mm, $l = 66$ mm, and $\psi = 18.86$ at $\vartheta_b = 226$ and 20 °C ($N = 0.72$ and 3.98, respectively).

results with Eqs. (43)–(45) and experimental data of [16] are presented. Analogous comparison with experimental data of [22] for the film and nucleate pool boiling with R113 on the cylindrical rod is shown in Fig. 9. The fin dimensions and values of n, N and ψ are given in the captions to these figures. The details of experiments are available in [16,22]. It can be seen that the predicted temperature distribution agrees well with experimental data.

5. Dimensionless geometrical and thermal characteristics of the straight plate and cylindrical pin fins

The varieties of the geometrical and thermal characteristics are used in numerous publications relating to the fin thermal analysis. An attempt is undertaken in the present study to relate such characteristics of fins with the power-law type dependence of the local heat transfer coefficient on the corresponding temperature difference between the fin surface and surrounding medium and to systematize these relations.

5.1. Geometrical characteristics of the SPF and CPF

Before to discuss the different thermal characteristics of the fins, the dimensionless geometrical fin characteristics and their relations are considered. First, the obvious relations between the reduced (half) profile area of the SPF (reduced volume of the CPF), its transverse dimension and height are expressed as

$$\widehat{A}_p = A_p/2 \equiv (a_p/2)(h_b/k)^2 = BiBi_l, \tag{46}$$

$$\widehat{V} = V/(4\pi) \equiv [v/(4\pi)](h_b/k)^3 = Bi^2Bi_l, \tag{47}$$

where $Bi = h_b(\delta/2)/k$ in Eq. (46) is a dimensionless half-thickness of the SPF, $Bi = h_b(r/2)/k$ in Eq. (47) is a dimensionless half-radius of the CPF, $Bi_l = h_b l/k$ is a dimensionless height of the SPF and CPF.

The fin aspect ratio $\psi = Bi_l/Bi$ and fin surface extension factor E_f are another dimensionless geometrical values used in the papers related to the fin analysis. In general case, the fin extension factor E_f is defined as the ratio of the fin surface area where the heat transfer occurs to the cross-sectional area of the fin, i.e. $E_f = F/A$. When the corresponding expressions for F and A is substituted into this formula, then with regard to Eq. (3) the following formula can be obtained for a fin with a non-insulated tip:

$$E_f = 1 + l/(\delta/2) = 1 + l/(r/2) = 1 + Bi_l/Bi = 1 + N/\sqrt{Bi} = 1 + \psi. \tag{48}$$

For a fin with an insulated tip the number one have to be omitted in all expressions of the last equation and superscript "*" after all variables must be introduced.

5.2. Thermal characteristics of the SPF and CPF

The different forms of the dimensionless thermal conductance of the fin, augmentation factor (effectiveness according to Gardner, see [5], sometimes named "removal number") and the fin efficiency will be considered below. These are the main thermal parameters of a fin in addition to the dimensionless temperature of the fin tip T_e and the temperature distribution along the fin height.

5.2.1. Fin base thermal conductances G_b and G_d

Dimensionless thermal conductance at the base of a fin with a non-insulated tip G_b is expressed by Eqs. (6) and (7) with T_e calculated by means of Eq. (23). When the heat transfer coefficient over the whole surface of a fin including its tip is uniform, the exponent $n = 0$ and upon substitution of the closed-form equation (16) into Eq. (7), one obtains

$$G_{b,n=0} = N\sqrt{1 - \frac{1-\omega^2 Bi}{\cosh N + \omega\sqrt{Bi}\sinh N}} \tag{49}$$

or, after the simple algebra,

$$G_{b,n=0} = N\frac{\tanh N + \omega\sqrt{Bi}}{1+\omega\sqrt{Bi}\tanh N}. \tag{50}$$

If a tip of a fin is insulated, then from the latter equation the known expression follows:

$$G_{b,n=0} = N\tanh N. \tag{51}$$

We introduced in [2] the relative base thermal conductance of a fin with an insulated tip. Accounting to the heat transfer from the fin tip, this expression changes to the following form:

$$\begin{aligned} G_{\rm d} &\equiv G_{\rm b}/G_{{\rm b},n=0} \\ &= (1+\omega\sqrt{Bi}\tanh N) \\ &\quad\times \frac{\sqrt{[2/(n+2)]\{1-T_{\rm e}^{n+2}[1-(n+2)\omega^2 Bi T_{\rm e}^n/2]\}}}{\tanh N + \omega\sqrt{Bi}}. \end{aligned} \tag{52}$$

The plots $G_{\rm b}$ and G_d *vs* N for $0 \leqslant N \leqslant 2$ and $-0.5 \leqslant n \leqslant 5$ are shown in Fig. 10a and b, respectively. It can be seen that the curves $G_{\rm b}$ and G_d *vs* N for various values of n and two extreme values of $\omega^2 Bi$ differ noticeably only for $N < 1$. When N approaches to 2 the curves for every value of n and $\omega^2 Bi = 0$ and $\omega^2 Bi = 0.015$ merge. For $N \geqslant 2$ these curves are valid for the above-mentioned "asymptotical" fins.

5.2.2. Base thermal conductance of the SPF (CPF) with given profile area (volume)

For the analysis of various fin optimization problems it is convenient to introduce the further kind of thermal conductance at a fin base, which is referred to as the thermal conductance at the specified fin volume $V = v(h/k)^3$ for the CPF or those at the specified profile area $A_{\rm p} = a_{\rm p}(h/k)^2$ for the SPF.

According to Eqs. (3), (46) and (47) the following relations are valid between the geometrical dimensions of a fin and its thermo-geometrical parameter N for the SPF and CPF, respectively:

$$\widehat{A}_{\rm p} \equiv A_{\rm p}/2 = NBi^{3/2}, \tag{53}$$

$$\widehat{V} \equiv V/(4\pi) = NBi^{5/2}. \tag{54}$$

Upon substitution of the expressions for the fin cross-sectional area A into the LHS of Eq. (7), one obtains

for the SPF

$$G_{\rm b} = \frac{g_{\rm b} l}{kA} = \frac{g_{\rm b}}{2zk}\frac{l}{\delta/2} = G_z\frac{Bi_l}{2Bi} = \widehat{G}_z\frac{Bi_l}{Bi} = \widehat{G}_z\psi, \tag{55}$$

for the CPF

$$G_{\rm b} = \frac{g_{\rm b} l}{kA} = \frac{g_{\rm b} h_{\rm b}}{k^2}\frac{l}{4\pi(r/2)^2} = G_{\rm c}\frac{Bi_l}{4\pi Bi^2} = \frac{\widehat{G}_{\rm c}}{Bi}\psi, \tag{56}$$

where $G_z = g_{\rm b}/(zk)$, $\widehat{G}_z = G_z/2$ and $G_{\rm c} = g_{\rm b}h_{\rm b}/k^2$, $\widehat{G}_{\rm c} = G_{\rm c}/(4\pi)$ are the dimensionless thermal conductances and reduced thermal conductances of the SPF and CPF, respectively.

Eq. (3) in combination with Eq. (55) for the SPF at $z \gg \delta$ gives

$$\psi \equiv \frac{Bi_l}{Bi} = \frac{N}{\sqrt{Bi}} = \frac{N^{4/3}}{\widehat{A}_{\rm p}^{1/3}}. \tag{57}$$

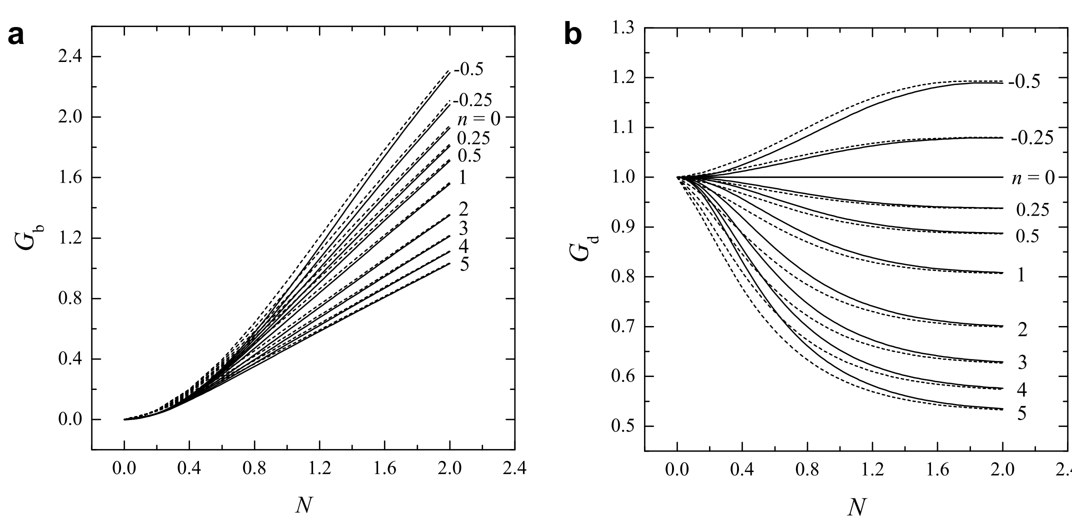

Fig. 10. Fin base thermal conductance $G_{\rm b}$ and related base thermal conductance $G_{\rm d}$ plotted as functions of the thermo-geometrical parameter N (a and b), respectively, for the SPF with an insulated tip ($\omega^2 Bi = 0$, solid lines) and non-insulated tip ($\omega^2 Bi = 0.015$, short-dashed lines) predicted by Eqs. (7), (52) and (23)–(26) at different values of n.

Then using Eq. (57), Eq. (55) can be rewritten as

$$G_b = \frac{\widehat{G}_z}{\widehat{A}_p^{1/3}} N^{4/3} = GN^{4/3}, \tag{58}$$

where the factor $\widehat{G}_z/\widehat{A}_p^{1/3}$ is denoted by G. Upon substitution of the expression for G_b from the RHS of Eq. (7) into Eq. (58) and resolving the obtained equation with respect to G, one can find, that

$$G \equiv \frac{\widehat{G}_z}{\widehat{A}_p^{1/3}} = \frac{\sqrt{[2/(n+2)]\{1 - T_e^{n+2}[1-(n+2)\omega^2 Bi T_e^n/2]\}}}{N^{1/3}}. \tag{59}$$

Similarly, for the CPF equation (3) in combination with Eq. (54) gives

$$\frac{Bi_l}{4\pi Bi^2} = \frac{N^{8/5}}{4\pi[V/(4\pi)]^{3/5}}. \tag{60}$$

Using denotations under Eq. (56), this equation can be rewritten as

$$G_b \equiv \frac{G_c}{4\pi[V/(4\pi)]^{3/5}} N^{8/5} = \frac{\widehat{G}_c}{\widehat{V}^{3/5}} N^{8/5} = GN^{8/5}, \tag{61}$$

where the factor $\widehat{G}_c/\widehat{V}^{3/5}$ is denoted by G. Upon substitution of the expression for G_b from the RHS of Eq. (7) into Eq. (61) and resolving this equation with respect to G, it is obtained

$$G \equiv \frac{\widehat{G}_c}{\widehat{V}^{3/5}} = \frac{\sqrt{[2/(n+2)]\{1 - T_e^{n+2}[1-(n+2)\omega^2 Bi T_e^n/2]\}}}{N^{3/5}}. \tag{62}$$

From Eqs. (46) and (47) follows that $\omega^2 Bi = \omega^2[Bi_a^2/(2N)]^{2/3}$ for the SPF and $\omega^2 Bi = \omega^2[Bi_v^3/(4\pi N)]^{2/5}$ for the CPF. After substitution of these values instead of ω^2Bi into RHS of Eqs. (59) and (62) these equations in combination with Eqs. (8)–(11) can be used for the optimization of the SPF and CPF by the numerical search of the position and the peak value of the function G *vs* T_e for the given parameters n and $\omega^2 Bi_a^{4/3}$ for the SPF as well as n and $\omega^2 Bi_v^{6/5}$ for the CPF. The different optimizations problems for the SPF and CPF will be comprehensively considered in the second part of this study [23].

Below we will consider the integral dimensionless thermal characteristics of a fin, specifically, fin augmentation factor K, fin efficiency η and their relations between each other and with another fin characteristics. We next will present all most important relations between any required characteristic from the list including four geometrical and three thermal characteristics and any other given characteristic from this list as well as the given generalized fin parameter(s) N and (or) G.

5.2.3. Fin augmentation factor (effectiveness) and fin efficiency: relations between fin parameters

The fin augmentation factor (ratio) K first referred to as "fin effectiveness" by Gardner in 1945 is defined in [5] as the ratio of the fin dissipation (equal, in the steady state, to the heat passing through the base of the fin by conduction) to the heat passing through the fin footprint of the base or prime surface if the fin not present. In the thermal conductance terms, used in this paper, the fin augmentation factor is defined as a ratio of the actual thermal conductance of a fin at its base to the thermal conductance of the prime surface with the area equal to the cross-sectional area of the fin at its base and with the same heat transfer coefficient as those at the fin base.

According to the LHS of Eqs. (55) and (56) the actual base thermal conductance of the SPF and CPF is expressed as $g_b = G_b kA/l$. The thermal conductance of the prime surface with the area equal to the cross-sectional area of the fin A and with the same heat transfer coefficient h_b as for the fin base is equal to $g_p = h_b A$. Dividing the former expression by the latter one, gives

$$K = g_b/g_p = G_b k/(h_b l) = G_b/Bi_l. \tag{63}$$

Upon substitution of the expressions G_b *vs* G and N (Eqs. (58) and (61) for the SPF and CPF, respectively) into Eq. (63), one obtains

for the SPF

$$K = \frac{G_b}{Bi_l} = \frac{GN^{4/3}}{Bi_l} = \frac{GN^{1/3}}{\sqrt{Bi}} = \frac{G}{N^{2/3}}\psi, \tag{64}$$

for the CPF

$$K = \frac{G_b}{Bi_l} = \frac{GN^{8/5}}{Bi_l} = \frac{GN^{3/5}}{\sqrt{Bi}} = \frac{G}{N^{2/5}}\psi. \tag{65}$$

Fin efficiency η is defined in [5] as the ratio of the actual heat dissipation of a fin to its ideal dissipation if the entire fin were at the same temperature as its base. In the thermal conductance terms used in present paper, fin efficiency is defined as the ratio of the actual thermal conductance of a fin at the base to the thermal conductance of this fin with uniform heat transfer coefficient over the entire fin surface. The heat transfer coefficient equal to h_b and the temperature excess over the whole fin equal to ϑ_b as so this fin has the infinite thermal conductivity of the fin material.

The base thermal conductance of the SPF and CPF with a non-insulated tip and infinite thermal conductivity of the fin material is equal to

$$g_{b,k=\infty} = h_b A E_f = h_b A(1 + N/\sqrt{Bi}) = h_b A(1+\psi). \tag{66}$$

Upon substitution of the function G_b *vs* G and N (Eqs. (58) and (61) for the SPF and CPF, respectively) into Eq. (63) and dividing the result by Eq. (66), one obtains the following formulae:

Table 1
The SPF required geometrical or thermal parameter as a function of its given parameter and fin parameter(s) N or (and) G

Given parameter	Required parameter						
	$\widehat{A}_p$	Bi	Bi_l	ψ	$\widehat{G}_z$	G_{zA_p}	K
$\widehat{A}_p$	$\widehat{A}_p$	$(\widehat{A}_p/N)^{2/3}$	$N^{2/3}\widehat{A}_p^{1/3}$	$(N^4/\widehat{A}_p)^{1/3}$	$G\widehat{A}_p^{1/3}$	$G/\widehat{A}_p^{2/3}$	$GN^{2/3}/\widehat{A}_p^{1/3}$
Bi	$N\mathrm{Bi}^{3/2}$	Bi	$N\mathrm{Bi}^{1/2}$	$N/\mathrm{Bi}^{1/2}$	$GN^{1/3}\mathrm{Bi}^{1/2}$	$G/(N^{2/3}\mathrm{Bi})$	$GN^{1/3}/\mathrm{Bi}^{1/2}$
Bi_l	Bi_l^3/N^2	$(\mathrm{Bi}_l/N)^2$	Bi_l	N^2/Bi_l	$G\mathrm{Bi}_l/N^{2/3}$	$GN^{4/3}/\mathrm{Bi}_l^2$	$GN^{4/3}/\mathrm{Bi}_l$
ψ	N^4/ψ^3	N^2/ψ^2	N^2/ψ	ψ	$GN^{4/3}/\psi$	$(G/N^{8/3})\psi^2$	$(G/N^{2/3})\psi$
$\widehat{G}_z$	$(\widehat{G}_z/G)^3$	$(\widehat{G}_z/G)^2/N^{2/3}$	$N^{2/3}(\widehat{G}_z/G)$	$N^{4/3}(G/\widehat{G}_z)$	$\widehat{G}_z$	$G(G/\widehat{G}_z)^2$	$GN^{2/3}(G/\widehat{G}_z)$
G_{zA_p}	$(G/G_{zA_p})^{3/2}$	$(G/G_{zA_p})/N^{2/3}$	$N^{2/3}(G/G_{zA_p})^{1/2}$	$N^{4/3}(G_{zA_p}/G)^{1/2}$	$G(G/G_{zA_p})^{1/2}$	G_{zA_p}	$GN^{2/3}(G_{zA_p}/G)^{1/2}$
K	$N^2(G/K)^3$	$N^{2/3}(G/K)^2$	$N^{4/3}(G/K)$	$N^{2/3}(K/G)$	$GN^{2/3}(G/K)$	$G(K/G)^2/N^{4/3}$	K

Table 2
The CPF required geometrical or thermal parameter as a function of its given parameter and parameter(s) N or (and) G

Given parameter	Required parameter						
	$\widehat{V}$	Bi	Bi_l	ψ	$\widehat{G}_c$	G_{cV}	K
$\widehat{V}$	$\widehat{V}$	$(\widehat{V}/N)^{2/5}$	$N^{4/5}\widehat{V}^{1/5}$	$(N^6/\widehat{V})^{1/5}$	$G\widehat{V}^{3/5}$	$G/\widehat{V}^{2/5}$	$GN^{4/5}/\widehat{V}^{1/5}$
Bi	$N\mathrm{Bi}^{5/2}$	Bi	$N\mathrm{Bi}^{1/2}$	$N/\mathrm{Bi}^{1/2}$	$GN^{3/5}\mathrm{Bi}^{3/2}$	$G/(N^{2/5}\mathrm{Bi})$	$GN^{3/5}/\mathrm{Bi}^{1/2}$
Bi_l	Bi_l^5/N^4	$(\mathrm{Bi}_l/N)^2$	Bi_l	N^2/Bi_l	$(G/N^{12/5})\mathrm{Bi}_l^3$	$GN^{8/5}/\mathrm{Bi}_l^2$	$GN^{8/5}/\mathrm{Bi}_l$
ψ	N^6/ψ^5	$(N/\psi)^2$	N^2/ψ	ψ	$GN^{18/5}/\psi^3$	$(G/N^{12/5})\psi^2$	$(G/N^{2/5})\psi$
$\widehat{G}_c$	$(\widehat{G}_c/G)^{5/3}$	$(\widehat{G}_c/G)^{2/3}/N^{2/5}$	$N^{4/5}(\widehat{G}_c/G)^{1/3}$	$N^{6/5}(G/\widehat{G}_c)^{1/3}$	$\widehat{G}_c$	$G(G/\widehat{G}_c)^{2/3}$	$GN^{4/5}(G/\widehat{G}_c)^{1/3}$
G_{cV}	$(G/G_{cV})^{5/2}$	$(G/G_{cV})/N^{2/5}$	$N^{4/5}(G/G_{cV})^{1/2}$	$N^{6/5}(G_{cV}/G)^{1/2}$	$G(G/G_{cV})^{3/2}$	G_{cV}	$GN^{4/5}(G_{cV}/G)^{1/2}$
K	$N^4(G/K)^5$	$N^{6/5}(G/K)^2$	$N^{8/5}(G/K)$	$N^{2/5}(K/G)$	$GN^{12/5}(G/K)^3$	$(G/N^{8/5})(K/G)^2$	K

for the SPF

$$\eta = \frac{GN^{1/3}}{\sqrt{Bi}E_\mathrm{f}} = \frac{K}{E_\mathrm{f}} = \frac{G}{N^{2/3}}\left(\frac{\psi}{1+\psi}\right), \tag{67}$$

for the CPF

$$\eta = \frac{GN^{8/5}}{N\sqrt{Bi}} = \frac{K}{E_\mathrm{f}} = \frac{G}{N^{2/5}}\left(\frac{\psi}{1+\psi}\right). \tag{68}$$

The fin base thermal conductance G in the RHS of Eqs. (67) and (68) is determined by Eqs. (59) and (62), respectively. For fins with an insulated tip the last therm in parentheses of the RHS of Eqs. (67) and (68) is equal to 1.

It is seen from Eqs. (67) and (68) that fin efficiency η is equal to the fin augmentation factor K divided by the fin surface extension factor E_f. Besides, the fin efficiency for the SPF and CPF with an insulated tip is equal to $\eta^* = G^*/N^{*2/3}$ and $\eta^* = G^*/N^{*2/5}$, respectively.

The formulae relating the required geometrical or thermal parameter of the SPF or CPF with any one of its given geometrical or thermal parameter, and the main generalized parameter(s) of the fin N and (or) G are presented in readily simple form in Tables 1 and 2 convenient for the use in practice. The thermal conductance G is used only in the cases when any one from the thermal parameters of the fin is given and (or) required.

6. Numerical example

The following numerical example is considered to illustrate the developed procedures taking into account for the geometrical and thermal characteristics of fins with insulated and non-insulated tips.

6.1. Input data

The next dimensionless characteristics of the SPF are given: exponent in Eq. (1) for nucleate pool boiling as a heat transfer mode $n = 2$, thermo-geometrical parameter $N = 1$, height $Bi_l = 0.1$, ratio of heat transfer coefficients on the tip and lateral surfaces of the fin $\omega = 1$. It is required to determine other dimensionless geometrical and thermal characteristics of this fin.

6.2. Solution procedure

(1) First, determine geometrical characteristics of the fin using the formulae collected in Table 1.Fin profile area is

$$A_\mathrm{p} = 2\widehat{A}_\mathrm{p} = 2Bi_l^3/N^2 = 2\cdot 0.1^3/1^2 = 2\times 10^{-3},$$

corresponding Biot number is

$$Bi_\mathrm{a} \equiv a_\mathrm{p}^{1/2}h_\mathrm{b}/k = \sqrt{A_\mathrm{p}} = \sqrt{2\times 10^{-3}} = 0.044721,$$

half-thickness of the fin, or the transverse Biot number is

$$Bi = Bi_l^2/N^2 = 0.1^2/1^2 = 0.01,$$

aspect ratio, or the fin height to half-thickness ratio is

$$\psi = N^2/Bi_l = 1^2/0.1 = 10,$$

fin extension factor we get according to Eq. (48),

$E_f = 1 + \psi = 1 + 10 = 11.$

(2) The preexponential factor μ is determined using Eq. (11) with given $\omega = 1$ and $Bi = 0.01$, obtained above,

$$\mu = 0.4(1 + 0.8\sqrt{Bi} - 0.414Bi) = 0.4(1 + 0.8\sqrt{0.01} - 0.414 \cdot 0.01) = 0.43034.$$

(3) The approximate dimensionless tip temperature $T^*_{e,app}$ of the fin with an insulated tip is determined by means of the procedure considered above. First, the coefficients P_1, P_2 and P_3 in Eq. (24) are calculated using Eq. (25)

$$P_1 = 3.7n/(1 + 3.55n) = 3.7 \cdot 2/(1 + 3.55 \cdot 2) = 0.91358,$$
$$P_2 = -2.24n/(1 + 4.62n + 0.62 \cdot 2^2) = -2.24 \cdot 2/(1 + 4.62 \cdot 2 + 0.62 \cdot 2^2) = -0.3522,$$
$$P_3 = 0.$$

Then $T^*_{e,app}$ is determined using Eq. (24):

$$T^*_{e,app} = \frac{\cosh(P_1 N + P_2 N^{1/2})}{\cosh(N)} = \frac{\cosh(0.91358 \cdot 1 - 0.3522 \cdot 1^{1/2})}{\cosh(1)} = 0.75288.$$

Next, the parameters Z and A_z are determined upon substitution of this $T^*_{e,app}$ value instead of T_e as well as the given values of n and N and calculated values of Bi and μ into the RHS of Eqs. (12) and (20), respectively

$$Z = NT_e^{\mu n} + \text{arcosh}\left(\frac{1}{\sqrt{1 - \omega^2 Bi}}\right) = 1 \cdot 0.75288^{0.43034 \cdot 2} + \text{arcosh}\left(\frac{1}{\sqrt{1 - 1 \cdot 0.01}}\right) = 0.88358,$$
$$A_Z = \mu n N T_e^{\mu n} \tanh Z = 0.43034 \cdot 2 \cdot 1 \cdot 0.7529^{0.43034 \cdot 2} \tanh(0.88358) = 0.47743.$$

The final T_e value is determined by the LHS of Eq. (23). Upon substitution of the calculated above $T^*_{e,app}$, Z, and A_z into the RHS of this equation, one obtains,

$$T_e = \frac{1 + A_Z}{\sqrt{1 - \omega^2 Bi}\cosh Z + (A_Z/T_e)} = \frac{1 + 0.47743}{\sqrt{1 - 1 \cdot 0.01}\cosh(0.88358) + (0.47743/0.75288)} = 0.723.$$

(4) The thermal conductance of the fin G is determined by substitution of the given values of n and N, and the calculated above values Bi and T_e into the RHS of Eq. (59),

$$G = \sqrt{\frac{2}{n+2}\left[1 - T_e^{n+2}\left(1 - \frac{n+2}{2}\omega^2 Bi T_e^n\right)\right]}/N^{1/3} = \sqrt{\frac{2}{2+2}\left[1 - 0.723^{2+2}\left(1 - \frac{2+2}{2} \cdot 1^2 \cdot 0.01 \cdot 0.723^2\right)\right]}\Big/1^{1/3} = 0.604.$$

(5) The thermal conductance G_z and specific thermal conductance G_{zA_p} of the SPF at base are determined by formulae presented in Table 1:

$$G_z = 2\widehat{G}_z = 2GBi_l/N^{2/3} = 2 \cdot 0.604 \cdot 0.1/1^{2/3} = 0.1208,$$
$$G_{zA_p} = GN^{4/3}/Bi_l^2 = 0.604 \cdot 1^{4/3}/0.1^2 = 60.4.$$

(6) Finally, the augmentation factor and efficiency of the SPF are determined using Table 1 and Eq. (67), respectively:

$$K = GN^{4/3}/Bi_l = 0.604 \cdot 1^{1/3}/0.1 = 6.04,$$
$$\eta = K/E_f = 6.04/11 = 0.5491.$$

6.3. Solution check

The parameter of the fin with uniform heat transfer coefficient over the whole fin surface ($n = 0$) and certain values of T_e and Bi calculated above is determined using Eq. (10),

$$N_0 = \ln\{[1 + \sqrt{1 - T_e^2(1 - Bi)}]/[T_e(1 + \sqrt{Bi})]\} = \ln\{[1 + \sqrt{1 - 0.723^2(1 - 0.01)}]/[0.723(1 + \sqrt{0.01})]\} = 0.7565.$$

The parameter of the considered fin with non-uniform heat transfer coefficient ($n = 2$) and certain values of N_0, T_e and μ calculated above is determined using Eq. (8),

$$N = N_0 T_e^{-\mu n} = 0.7565 \cdot 0.723^{-0.43034 \cdot 2} = 1.0001,$$

that coincides with the given N value to the three digit precision.

If n, N, and the thermal parameter of the SPF G_z are specified, then the characteristics of the fin with an insulated tip are determined first. The value of $T^*_{e,app} = 0.75288$ is already determined above. The thermal conductance G^* of the fin with an insulated tip is calculated by means of Eq. (59) at $\omega = 0$ in the RHS

$$G^* = \sqrt{(2/4)(1 - 0.75288^4)}/1^{1/3} = 0.58254.$$

Then the transverse Biot number Bi^* is determined by formula presented in Table 1:

$$Bi^* = (\widehat{G}_z/G^*)^2/(N^{2/3}) = (G_z/G^*)^2/(4N^{2/3}) = (0.1208/0.58254)^2/(4 \cdot 1^{2/3}) = 0.01075.$$

The preexponential factor μ is determined using Eq. (11):

$$\mu = 0.4(1 + 0.8\sqrt{0.01075} - 0.414 \cdot 0.01075) = 0.4314.$$

The parameters Z and A_z are determined by means of Eqs. (12) and (20), respectively

$$Z = 1 \cdot 0.75288^{0.4314 \cdot 2} + \operatorname{arcosh}(1/\sqrt{1 - 0.01075}) = 0.88684,$$
$$A_z = 0.4314 \cdot 2 \cdot 1 \cdot 0.75288^{0.4314 \cdot 2} \tanh(0.88684) = 0.47941.$$

The dimensionless temperature excess T_e of this fin with a non-insulated tip is determined using Eq. (23):

$$T_e = \frac{1 + 0.47941}{\sqrt{1 - 0.01075}\cosh(0.88684) + (0.47941/0.75288)} = 0.72208.$$

This value is less than the value 0.723, obtained above, by as little as 0.13%. Upon substitution of this value T_e in the RHS of Eq. (59) one obtains the thermal conductance G of this fin with non-insulated tip:

$$G = \sqrt{(2/4)\{1 - 0.72208^4[1 - (4/2)0.01075 \cdot 0.72208]\}/1^{1/3}} = 0.60513.$$

Then the transverse Biot number Bi is determined using the formula presented in Table 1:

$$Bi = (G_z/G)^2/(4N^{2/3}) = (0.1208/0.60513)^2/(4 \cdot 1^{2/3}) = 0.00996.$$

This value is less than the exact one, obtained above, by as little as 0.4%.

Any required characteristic of the CPF may also be determined by the same procedures if n, ω, N and any one of the geometrical or thermal parameters of this fin are given.

7. Conclusions

(1) The effect of the fin tip heat loss is taken into account both for the inverse closed-form and the direct recurrent solutions.
(2) The inverse closed-form solution for fins with non-insulated tips is derived in the same form of Eq. (8) as for fins with insulated tips except that both the pre-exponential coefficient μ and thermo-geometrical parameter N_0 of a fin with $n = 0$ depend, in addition, on the complex parameter $\omega^2 Bi$.
(3) The obtained closed-form inverse solution is inverted into the direct recurrent solution to determine T_e and the temperature distribution along a fin height for the given values of n, N, and $\omega^2 Bi$.
(4) The linearization method, developed in [2], is used to transform the obtained recurrent solutions with a poor convergence rate both for T_e and for the temperature distribution along a fin height into the recurrent formulae with a very high convergence rate.
(5) The correlation $T^*_{e,app}$ *vs* N and n obtained for a fin with an insulated tip only is used instead of T_e in the RHS of the direct recurrent solution Eq. (23) with high convergence rate to transform it into the direct closed-form solution for fins with non-insulated and, particularly, insulated tips.
(6) The direct closed-form formula based on the recurrent one is derived also for the temperature distribution along a fin with a non-insulated tip.
(7) The formulae were systematized, which allow using the given geometrical or thermal parameter of a fin and given or calculated main dimensionless fin parameters N and G to determine all other dimensionless geometrical and thermal parameters of the SPF and CPF, for example, fin height and thickness (radius), aspect ratio, different types of the thermal conductance, efficiency, augmentation factor, *etc*. These formulae collected in Tables 1 and 2 can be used also for the solution of the fin optimization problems. Such problems will be considered more detailed in the second part of this study.

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